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The University of Melbourne
Semester 2 Assessment 2004

Department of Mathematics and Statistics

620-302 Chance and Options Pricing

Reading time: fifteen (15) minutes
Writing time: three (3) hours
This paper has five (5) pages

Authorised materials:

Hand-held calculators may be used.

Instructions to invigilators:

No books or lecture notes are allowed in the examination room.
This paper is to remain in the examination room.

Instructions to students:

There are ten (10) questions.
All questions may be attempted.
They carry weights as shown in the brackets after the question statement.
The total marks available is 100.
Working and/or reasoning must be given to obtain full credit.

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1. Consider a three-period binomial financial market with $u = 4/3$, $d = 2/3$ and $r = 0$ (i.e. the prices have already been discounted). Let the current (time $t = 0$) stock price be $S_0 = 81$.

   (a) Specify an appropriate outcome space Ω describing the three-period binomial market. Find the risk-neutral probability $\mathbf{P}^*$ on this Ω (that is, give the risk-neutral probabilities $\mathbf{P}^*(\{\omega\})$ for all outcomes $\omega \in \Omega$).

   (b) Use the diagram method to price a European call option (on one share of the stock) with maturity $T = 3$ and strike $K = 88$.

   (c) What is the price of the call option from part (b) at time $t = 2$ if the stock price $S_2 = 72$?

   (d) For the call option from part (b), find the replicating portfolio (Δ_3, b_3) for the third time period if $S_2 = 72$. How is the time $t = 2$ value of the portfolio related to your answer in part (c)?

   (e) State the put-call parity and use it to obtain the time $t = 0$ arbitrage-free price of a European put option (on one share of the stock) with the same expiry and strike as the call option from part (b).

   $[2 + 3 + 1 + 4 + 2 = 12]$

2. (a) Explain what is meant by a *complete arbitrage-free financial market.*

   (b) State a necessary and sufficient mathematical condition for a market to be arbitrage-free and complete. Introduce whatever notations and assumptions you need for that.

   $[3 + 4 = 7]$

3. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with a characteristic function $\varphi(t) = \exp\{-|t|^{2/3}\}$. For each of the statements below, say if it is TRUE or FALSE, and explain your answers, referring to the basic properties of characteristic functions or doing necessary calculations.

   (a) The distribution of X_1 is infinitely divisible.

   (b) The distribution of X_1 is symmetric (about the origin).

   (c) $\mathbf{E}\,X_1 = 0$.

   (d) The distribution of $(X_1 + \cdots + X_n)/\sqrt{n}$ converges to a normal one as $n \to \infty$.

   (e) The distribution of X_1 has got a differentiable density.

   (f) There is a constant $c < 0$ such that the distribution of $X_1 + 8X_2$ coincides with that of cX_1.

   $[2 + 2 + 2 + 2 + 2 + 3 = 13]$

4. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathbf{E}\, X_1 = 0$ and $\mathbf{E}\, X_1^2 = 1$. Put $S_0 = 0$, $S_n = S_{n-1} + X_n$, $n \ge 1$.

   (a) Find the best in mean quadratic predictor for S_{12} from S_{10}.

   (b) Using the properties of conditional expectations or otherwise, compute the conditional expectations $\mathbf{E}\,(X_1 | \, S_{10})$ and $\mathbf{E}\,(X_1 | \, S_1, S_2, \ldots, S_{10})$.

   (c) Compute the conditional expectation $\mathbf{E}\,(S_{n+m}^2 | \, S_n)$, $n, m \ge 0$.

   (d) Give the definition of a martingale in discrete time and show that the process $Z_n = S_n^2 - n$, $n = 0, 1, 2, \ldots$ satisfies the definition.

   Hint. (b) Note that $\mathbf{E}\,(X_1 | \, S_{10}) = \mathbf{E}\,(X_2 | \, S_{10})$ due to symmetry.

   [2 + 3 + 2 + 3 = 10]

5. It is assumed that, under the equivalent martingale measure, the stock price in the Black-Scholes model evolves according to the process

$$S_t = 10 \exp\{0.06t + 0.2W_t\}, \quad t \ge 0, \tag{1}$$

   where $\{W_t\}_{t \ge 0}$ is a standard Brownian motion process and the time unit is one year.

   (a) Explain, in your own words, what the term "equivalent martingale measure" means.

   (b) Give the volatility parameter σ and the spot interest rate r for the model in equation (1).

   (c) Verify by a direct computation that the discounted stock price process is a martingale. [*Hint:* For $X \sim N(0, s)$, one has $\mathbf{E}\, e^{uX} = e^{u^2 s/2}$.]

   (d) Write down the payoff function of a European put option (on one share of the stock) with maturity $T = 1$ and strike $K = 11$.

   (e) Write down the time t price of the put from part (d) in the expectation form and demonstrate how to compute it when $t = 0$ (you don't need to finish the calculation—just outline its main steps).

   (f) Compute the stochastic differential of the process $\{S_t\}$ and hence derive a stochastic differential equation for the process. Explain briefly why $\{S_t\}$ is a diffusion and identify its drift and diffusion coefficients.

   (g) Let $\tau_1 = \min\{t \ge 0 : S_t \ge 11\}$ and $\tau_2 = \min\{t \ge 0 : S_t \le 9\}$. Explain why τ_j, $j = 1, 2$, are stopping times for the process, and then give a mathematical argument showing that $\tau = \min\{\tau_1, \tau_2\}$ is also a stopping time.

   (h) Use the Optional Stopping Theorem (without verifying its conditions) for an appropriate martingale of the Brownian motion to compute $\mathbf{E}\,\tau_1$. Using the same theorem, what could you deduce about the value of $\mathbf{E}\,\tau_2$?

   [1 + 2 + 2 + 1 + 3 + 3 + 2 + 3 = 17]

6. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with a mean $\mathbf{E}\, X_1 = 2$. Use the method of characteristic functions to show that

$$Y_n = n^{-2}(X_1 + 2X_2 + 3X_3 + \cdots + nX_n) \xrightarrow{d} 1 \quad \text{as } n \to \infty$$

(that is, similarly to the statement of the Law of Large Numbers, the random variables Y_n converge in distribution to a constant).

Hint. Recall that, if $\varphi(t) = \mathbf{E}\, e^{itX}$ is the characteristic function of a random variable X with a finite mean $\mu = \mathbf{E}\, X$, then, for any fixed t,

$$\ln \varphi(\varepsilon t) = \ln[1 + (\varphi(\varepsilon t) - 1)] = i\mu\varepsilon t + o(\varepsilon) \quad \text{as } \varepsilon \to 0$$

(where, as usual, $o(\varepsilon)$ denotes a quantity with the property $o(\varepsilon)/\varepsilon \to 0$), and that $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$.

[4]

7. Suppose that the interest rate is modelled using the following stochastic differential equation (SDE):

$$dX_t = (r - X_t)dt + dW_t, \quad t \geq 0, \quad X_0 = r/2, \tag{2}$$

where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion process.

(a) Use the rules of Itô calculus to verify that $X_t = r + e^{-t}\left(\int_0^t e^s dW_s - r/2\right)$ satisfies the SDE and the initial condition from (2).

(b) Compute the mean and the variance of X_t for a given $t > 0$.

(c) By quoting an appropriate result, give the distribution of X_t for a given t.

(d) Write down the forward and backward Kolmogorov equations for the process $\{X_t\}$.

(e) One can show that the process $\{X_t\}$ has a stationary density $\pi(y)$, $y \in \mathbf{R}$. Use the result of part (d) to write down an ordinary differential equation for $\pi(y)$. Solve the equation and hence derive the stationary distribution of the process.

(f) Find the limit (as $t \to \infty$) of the distribution of X_t you found in part (c) and hence verify your answer to part (e).

[3 + 2 + 2 + 2 + 2 + 1 = 12]

8. A diffusion process $\{X_t\}_{t\geq 0}$ is given by the stochastic differential equation (SDE)

$$dX_t = -tX_t dt + X_t dW_t, \quad t \geq 0,$$

with a non-random initial condition $X_0 = x_0$ (as usual, $\{W_t\}_{t\geq 0}$ is a standard Brownian motion process).

(a) Use the SDE to obtain an ordinary differential equation for $m_t = \mathbf{E}\, X_t$ and hence derive the mean function m_t by solving the equation.

(b) Show that the process $Y_t = X_t^2$ satisfies the SDE

$$dY_t = 2(1-t)Y_t\, dt + 2Y_t\, dW_t.$$

(c) Use the SDE from part (b) to show that, if the process $\{X_t\}_{t\geq 0}$ starts at $X_0 = 0$, then it remains there forever: for any $t > 0$, $X_t = 0$ with probability one.

Hint. (c) Recall that if $\mathbf{E}\, Z^2 = 0$ for a random variable Z, then $Z = 0$ with probability one.

[4 + 2 + 4 = 10]

9. Let $X_t = X_0 + W_t$, where X_0 is an initial value, independent of the standard Brownian motion process $\{W_t\}_{t\geq 0}$. For $a < 0 < b$, denote by

$$\tau = \min\{t > 0 : X_t = a \text{ or } X_t = b\}$$

the first time the process $\{X_t\}_{t\geq 0}$ takes one of the two values a or b.

(a) State the ordinary differential equations (no need to derive them!) for the functions

$$V(x) = \mathbf{P}\,(X_\tau = b \,|\, X_0 = x) \quad \text{and} \quad U(x) = \mathbf{E}\,(\tau \,|\, X_0 = x), \quad x \in (a, b),$$

and also state appropriate boundary conditions for the equations. Solve the equations to find the functions $V(x)$ and $U(x)$.

(b) Verify your answers to the last question in part (a) above using the Optional Stopping Theorem for the martingales $\{X_t\}$ and $\{Y_t = (X_t - x)^2 - t\}$ (assuming that $X_0 = x$).

[6 + 3 = 9]

10. (a) Give a definition of the Brownian bridge process $\{X_t\}_{0\leq t\leq 1}$.

(b) Compute (i) the mean function $m_t = \mathbf{E}\, X_t$ and (ii) the covariance function $\gamma_{s,t} = \mathrm{Cov}\,(X_s, X_t)$, $0 \leq s \leq t \leq 1$, of the Brownian bridge process.

[3 + 3 = 6]