The University of Melbourne

Semester 1 Assessment — June, 2005

Department of Mathematics and Statistics

620-201 Probability

Exam Duration: 3 Hours

Reading Time: 15 Minutes

This paper has 5 pages

Authorised materials:
Hand-held electronic calculators (including graphics calculators) may be used.

Instructions to Invigilators:
Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
Students may take this paper with them at the end of the exam.

Instructions to Students:
This paper has **ten** (10) questions.
Attempt as many questions, or parts of questions, as you can.
Questions carry marks as shown in the brackets after the question statement.
The total of marks available for this examination is 100.
Working and/or reasoning must be given to obtain full credit.

1. Consider a random experiment with sample space Ω.

   (a) Write down the axioms which must be satisfied by a probability mapping P defined on the events of the experiment.

   (b) Using the axioms, show that,

      (i) $P(\emptyset) = 0$,

      (ii) for any event A, $P(A^c) = 1 - P(A)$,

      (iii) for any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

   (c) (i) State what it means for two events A and B in a random experiment to be independent.

      (ii) Show that if A and B are disjoint and independent, and if $P(B) > 0$, then $P(A) = 0$.

[13 marks]

2. (a) Let A and B be events in a random experiment. State Bayes' Formula for the probability $P(B|A)$.

   (b) An employment agency uses a test to decide if applicants are suitable for a particular job. Suppose that if a person is suitable there is a probability of 0.8 that the test will confirm this. If a person is not suitable, there is still probability of 0.15 that the test will say that they are suitable. Suppose also that the proportion of individuals in the whole population who are suitable for the job is 1 in 25.

      (i) What is the probability that an arbitrary person who the test says is suitable for the job actually is suitable?

      (ii) In the light of your answer to part (i), comment on the usefulness of the test in selecting employees.

[5 marks]

3. In the Australian gambling game of "two-up", two fair coins are tossed. If they are both heads or both tails, then the game is resolved and people who have bet on "heads" or "tails" (whichever comes up) collect their winnings (if you win a \$1 bet you get back your original bet plus an extra \$1, that is \$2). However, if there is one head and one tail (that is, *"odds"* come up), the game is not resolved and the coins have to be tossed again. In a series of tosses of two coins, let N be the number of times that "odds" comes up before either "heads" or "tails" comes up.

   (a) Giving your reasons, name the distribution of the random variable N and give the value of any parameter(s).

   (b) What the probability mass function of N?

   (c) What is the expected value of N?

   (d) In casinos, if "odds" comes up five times in a row, then the game is aborted and the casino collects all bets that have been laid on either "heads" or "tails". What is the probability of this event?

(e) Assume that you bet \$10 on "heads". What is the expected value of the amount that you receive back from the casino?

[9 marks]

4. (a) Consider the random variable $X \stackrel{d}{=} \exp(\lambda)$.
   (i) What is the range of possible values of the random variable $Y = e^{-\lambda X}$?
   (ii) What is the distribution of Y?
   (iii) Hence, or otherwise, explain how you would generate an observation on X from an observation on $U \stackrel{d}{=} R(0,1)$.

   (b) The random variable $X \stackrel{d}{=} R(-2,2)$ (that is X has a continuous uniform distribution over the interval $(-2,2)$). If $Y = X^2$, find the distribution function $F_Y(y)$ of Y. (In your answer be careful to give expressions for all values of $y \in \mathbb{R}$).

[9 marks]

5. (a) Consider independent discrete random variables X and Y with probability mass functions $p_X(x)$ and $p_Y(y)$ respectively. Let S_X be the set of possible values of X. Using the Law of Total Probability and justifying all your steps, derive the expression

$$p_Z(z) = \sum_{x \in S_X} p_X(x) p_Y(z-x)$$

   for the probability mass function $p_Z(z)$ of $Z = X + Y$.

   (b) Hence, or otherwise, derive the probability mass function $p_Z(z)$ for $Z = X + Y$ where $X \stackrel{d}{=} \text{Pn}(\mu)$ and $Y \stackrel{d}{=} \text{Pn}(\lambda)$.

[7 marks]

6. Let X and Y have joint probability density function

$$f_{(X,Y)}(x,y) = \begin{cases} kx & 0 < x < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Find k.
   (b) Explain why X and Y are dependent (no calculations should be required).
   (c) Find the marginal probability density functions of X and Y.
   (d) For a fixed $y \in (0,1)$ write down the set of values of x for which $f_{X|Y=y}(x|y)$ is non-zero.
   (e) Find $f_{X|Y=y}(x|y)$ over the range that you derived in (d).
   (f) Find $E(X|Y)$.

(g) Find $P(X < Y)$.

(h) Assume that someone who only knows the marginal densities mistakenly assumes that X and Y are independent. What value will they obtain for $P(X < Y)$?

[14 marks]

7. (a) Consider the following two stage random experiment. In the first stage we choose a coin at random from an urn. The urn contains equal numbers of two types of biased coins whose probabilities of coming up heads equal 1/4 and 3/4 respectively. In the second stage we toss the chosen coin n times and count X, the number of heads. Let H be a random variable whose value is the probability that the coin chosen in stage 1 comes up heads, so H can take the two values 1/4 and 3/4.

   (i) Write down the pmf $p_H(h)$ of H and hence find $E(H)$, $E(H^2)$ and $V(H)$.

   (ii) Showing all your steps, find the random variables $E(X|H)$ and $V(X|H)$.

   (iii) Hence, or otherwise, find $E(X)$ and $V(X)$.

   (b) Now consider replacing the urn in Stage 1 of the experiment by a device which produces a coin with (random) probability of coming up heads $H \stackrel{d}{=} R(0,1)$ ie uniformly and continuously distributed on the interval $(0,1)$.

   (i) Find the conditional pgf $P_{X|H}(z)$

   (ii) Hence, or otherwise, find the pgf $P_X(z)$. Name the distribution of X and give the value of any parameter(s).

[14 marks]

8. Consider two independent, non-negative, integer valued random variables X and Y, with probability generating functions $P_X(z)$ and $P_Y(z)$ respectively. Show that the probability generating function of $W = X + Y$ is given by $P_W(z) = P_X(z)P_Y(z)$.

[5 marks]

9. In this question, you may use the fact that, for $s > 0$,

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} du. \qquad (*)$$

For non-negative integer r, a random variable X that has *the Chi-square distribution with r degrees of freedom* has a probability density function

$$f(x) = \frac{x^{r/2-1} e^{-x/2}}{\Gamma(r/2) 2^{r/2}}.$$

   (a) For what values of t is the moment generating function $M_X(t)$ of such a random variable X defined?

(b) Derive $M_X(t)$. (Hint: You will need to use an appropriate integral substitution to reduce the integral to something that looks like the right hand side of (*)).

(c) Calculate the mean and variance of X.

[7 marks]

10. (a) Consider the Branching Process $\{X_n, n = 0, 1, 2, 3, \ldots\}$ where X_n is the population size at the nth generation. Assume $P(X_0 = 1) = 1$ and that the pgf of the common offspring distribution N is

$$A(z) = \frac{1}{3 - 2z}$$

(i) Express $A(z)$ as a power series and hence find $P(N = 6)$.

(ii) If $q_n = P(X_n = 0)$ for $n = 0, 1, \ldots$, write down an equation relating q_{n+1} and q_n. Hence, or otherwise, evaluate q_n for $n = 0, 1, 2$.

(iii) Find the extinction probability $q = \lim_{n\to\infty} q_n$.

(b) The weather in Markovograd is a discrete time Markov chain on the states $\{1 = \text{'perfect'}, 2 = \text{'average'}, 3 = \text{'miserable'}\}$ and with transition probability matrix P given by

$$\begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.5 & 0.4 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}$$

If Thursday is 'average' what is the probability that

(i) Saturday is 'perfect',

(ii) Sunday is 'perfect',

(iii) both Saturday and Sunday are 'perfect'.

[16 marks]

End of the exam