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The University of Melbourne
Semester 2 Assessment 2005

Department of Mathematics and Statistics

620-302 Chance and Options Pricing

Reading time: fifteen (15) minutes
Writing time: three (3) hours
This paper has four (4) pages

Authorised materials:

Hand-held calculators may be used.

Instructions to invigilators:

No books or lecture notes are allowed in the examination room.
This paper is to remain in the examination room.

Instructions to students:

There are seven (7) questions.
All questions may be attempted.
They carry weights as shown in the brackets after the question statement.
The total marks available is 100.
Working and/or reasoning must be given to obtain full credit.

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1. (a) Write down the payoff function of a European call option (on one share of the stock). Sketch a plot of the payoff as a function of the stock price S_T at maturity assuming the strike is equal to $K_1 = 1$.

   (b) Write down the payoff function of a European put option (on one share of the stock). Sketch a plot of the payoff as a function of the stock price S_T at maturity assuming the strike is equal to $K_2 = 2$.

   (c) A *bull spread* is a portfolio formed by buying a call with a strike price K_1 and selling a call with a strike price $K_2 > K_1$. Both calls are on the same underlying stock and have the same expiry date T. Plot the payoff function of a bull spread with $K_1 = 1$, $K_2 = 2$ as a function of the stock price S_T at maturity.

   (d) Which option is more expensive: a bull spread with strikes $K_2 > K_1$ or a call with the strike K_1 (on the same number of shares of the same underlying stock, with the same expiry date)? Explain.

$[3 + 3 + 3 + 3 = 12]$

2. Consider a three-period binomial financial market with $u = 1.2$, $d = 0.9$ and $r = 0.1$. Let the current (time $t = 0$) stock price be $S_0 = 1000$.

   (a) Compute the risk-neutral probability of the event $\{S_1 = 1200\}$.

   (b) Explain what is meant by a *complete arbitrage-free financial market.* Say if the above-described binomial market is arbitrage-free and explain why (you may wish to refer to a necessary and sufficient condition for a binomial market to be arbitrage-free).

   (c) Explain what is meant by a *replicating portfolio* for a contingent claim and how the value of the portfolio is related to the price of the claim.

   (d) Use the diagram method to find the time $t = 0$ price of a European put option (on one share of the stock) with maturity $T = 3$ and strike $K = 1200$.

   (e) What is the price of the put option from part (d) at time $t = 1$ if the stock price $S_1 = 900$?

   (f) State the put-call parity and use it to obtain the time $t = 0$ arbitrage-free price of a European call option (on one share of the stock) with the same expiry and strike as the put option from part (d).

$[2 + 4 + 4 + 4 + 2 + 2 = 18]$

3. Let X and Y be independent random variables, each distributed uniformly on the interval $[0, 1]$.

   (a) Compute the characteristic function (ch.f.) of X.

   (b) Use the general properties of ch.f.'s to find the ch.f. of $V = 2(X - Y)$.

   (c) Compute the conditional expectation $\mathbf{E}\,(V^2|X)$.

$[2 + 4 + 4 = 10]$

4. Let X and Y be random variables on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, $\varphi(t)$ the characteristic function of X, and let $\mathbf{E}|X| < \infty$.

For each of the following statements, say whether it is **TRUE** or **FALSE** and explain your answers by quoting basic properties and/or theorems discussed in the subject, or doing necessary calculations.

(a) One always has $\varphi(0) = 1$.

(b) One always has $\varphi(t) = \varphi(-t)$ for all $t \geq 0$.

(c) If $\int_{-\infty}^{\infty} |\varphi(t)|dt < \infty$, then the distribution of X has a continuous density.

(d) One always has $\mathbf{E}(X|\mathcal{F}) = X$.

(e) One always has $\mathbf{E}(X|Y) = \mathbf{E}(X| - Y)$.

(f) For any σ-algebras $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ one always has

$$\mathbf{E}(X|\mathcal{F}_2) = \mathbf{E}\left[\mathbf{E}(X|\mathcal{F}_1)\middle| \mathcal{F}_2\right].$$

(g) For any σ-algebras $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ one always has

$$\mathbf{E}(X|\mathcal{F}_1) = \mathbf{E}\left[\mathbf{E}(X|\mathcal{F}_1)\middle| \mathcal{F}_2\right].$$

(h) If $\mathbf{E}X^2 < \infty$, then the random variables $V_1 = X - \mathbf{E}(X|Y)$ and $V_2 = \mathbf{E}(X|Y)$ are uncorrelated:

$$\mathrm{Cov}(V_1, V_2) \equiv \mathbf{E}V_1V_2 - \mathbf{E}V_1 \cdot \mathbf{E}V_2 = 0.$$

(i) If $\{X_n\}_{n\geq 1}$ is a martingale and τ a stopping time for it with $\mathbf{E}\tau < \infty$, then always $\mathbf{E}X_\tau = \mathbf{E}X_0$.

(j) If τ_1 and τ_2 are stopping times for the martingale from part (i), then the random variable $\tau = \max\{\tau_1, \tau_2\}$ is also a stopping time for $\{X_n\}_{n\geq 1}$.

[2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 20]

5. Denote by $\{N_t\}_{t\geq 0}$ a Poisson process with a unit rate. Show that the following two processes:

(a) $X_t = N_t - t, \ \ t \geq 0$,

(b) $Y_t = (N_t - t)^2 - t, \ \ t \geq 0$,

are both martingales with respect to the filtration $\mathcal{F}_t = \sigma(N_s, \ s \leq t)$, $t \geq 0$.

Hint: If Z is a Poisson random variable with parameter λ, then $\mathbf{E}Z = \mathrm{Var}(Z) = \lambda$.

[4 + 4 = 8]

6. Suppose that $\{S_t\}_{t\geq 0}$, $S_0 = 5$, is the stock price process in the Black-Scholes model, and assume a 10% volatility ($\sigma = 0.1$) and a spot interest rate $r = 0.05$.

   (a) Explain, in your own words, what the term "equivalent martingale measure" means.

   (b) Write down a representation for the price process in the form of a geometric Brownian motion process under the equivalent martingale measure.

   (c) Verify by a direct computation that the discounted stock price process (use the result of part (b)) is a martingale. [*Hint:* The moment generating function of a normal random variable $Z \sim N(0, a^2)$ is equal to $\mathbf{E}\, e^{uZ} = e^{a^2u^2/2}$.]

   (d) Write down the time t price of a European call option (on one share of the stock) with maturity T and strike K in the expectation form.

   (e) Compute the time $t = 0$ price of a European call with maturity $T = 1$ and strike $K = 6$ (if you don't remember the Black-Scholes formula, derive it using the result of part (d); to compute the values of the standard normal c.d.f. $N(h)$, you may wish to use linear interpolation and the following approximate values: $N(-1.2) = 0.1151$, $N(-1.3) = 0.0968$, $N(-1.4) = 0.0808$, $N(-1.5) = 0.0668$).

$[3 + 3 + 3 + 3 + 4 = 16]$

7. A diffusion process $\{X_t\}_{t\geq 0}$ is given by the stochastic differential equation

$$dX_t = -\alpha X_t\, dt - \sqrt{1 - X_t^2}\, dW_t, \quad X_0 = x \in (-1, 1); \quad \alpha \text{ is a fixed real number.}$$

   (a) Compute the stochastic differential of the process $Y_t = \frac{1}{2}\, t^2 X_t^2$, $t \geq 0$.

   (b) Write down the backward and forward Kolmogorov equations for the process $\{X_t\}_{t\geq 0}$.

   (c) One can show that the process $\{X_t\}_{t\geq 0}$ is *ergodic* when $\alpha > 0$. State an ordinary differential equation for the stationary density $\pi(y)$, $y \in (-1, 1)$, of the process and comment on how this equation relates to the result of part (b). Solve the equation and hence derive the stationary density of the process (it suffices to give the answer in the form $\pi(y) = Cf(y)$, where $f(y)$ is given explicitly, and C is a constant — no need to compute the latter!).

   (d) Let $-1 < a < b < 1$. State an ordinary differential equation (no need to derive it!) for the function

$$V(x) = \mathbf{P}\,(X_\tau = b \,|\, X_0 = x), \quad a < x < b,$$

   where $\tau = \min\{t \geq 0 : X_t = a \text{ or } X_t = b\}$ is the first time the process $\{X_t\}_{t\geq 0}$ hits one of the barriers a and b. Verify that, in the case when $\alpha = 1$, the differential equation will be solved by $V(x) = C_1\big[\ln(1 + x) - \ln(1 - x) + C_2\big]$. Specify boundary conditions for the differential equation and use them to find the constants C_1 and C_2.

$[4 + 4 + 4 + 4 = 16]$

The end of the examination paper